THE DERIVATIVE OF AN IDENTITY

WITH THE ANALYSIS OF FUNCTIONS OF VECTOR STATES

WARNING: Do not try to create a Geometric picture while studying this material. The depth of the Mathematics is constructed from a rigorous application of the Hermitian CONJUGATE VARIABLE, and NOT functions of two variables.

PREREQUISITE

The Mathematics in this text is of High Order Source(Advanced). Its beyond High School, and Tertiary. An Adequate background on the concepts of the following is desired:
Characteristic Polynomial Function
Differentiation Rules
Sequences and Series
Partial Derivatives
Linear Approximation
The Average Value of a Function
Standard Equation of the Circle with Radius (n)
Vectors
Complex Numbers
Tensor Calculus
Pauli Matrices
Integration
Programming Language

PREFACE

It is a bad habit for Physicists to take their most successful abstractions to be real properties of the world - Nathaniel Mermin

To the Mathematician, Computer Programmer, Electrical Engineer and Physicist,

Understanding the concept of an Absolute Value of State Space, is the most gratifying experience, and an achievement that takes lots of intense study and patience.
The title of my previous book [Rational Commensurability and Why Division by Zero Is Not Possible] pointed on the properties of the number 0; while this text now REFLECTS the number 1. I must admit the structure of both materials are complete that I could not add or subtract any formalism to their architecture.

The depth of Mathematical maturity in visualizing its core, is the motivation for the title, The Derivative of An Identity.
I'm most humbled to be able to clarify a well and thoroughly established principle.

INTRODUCTION:

In this text, we'll have a a rigorous and an analytic treatment of the function of a vector state in space. We'll build the concept using the properties of the following generalized topics:

★ **Linear Operators : That the $\sqrt{-1} = i$, $-i * i = 1$ and $i^2 = -1$**

- That Polyphase AC is a Mathematical Model with NO INDEPENDENT COMPONENTS; only exists as SQUARES or PRODUCTS.
- Elementary Algebra of Complex Quantities: First, that the EFFECTIVE VALUE of a Sine Wave = 0.

DC is represented by the following relation : $Y = \sqrt{b^2 + g^2}$ and $Z = \sqrt{r^2 + x^2}$
Complex conjugate of AC is represented by the following relation : $Y = g + ib$ and $Z = r - ix$
Ohms Law is characterized by the relation $E = IZ$ and $I = EY$ where
Z = Impendance = $\frac{1}{Y}$ which is an absolute value derived from the equation $Y^2 = b^2 + g^2$ and
Y = Admittance = $\frac{1}{Z}$ which is an absolute value derived from the equation $Z^2 = r^2 + x^2$
Proof of Z and Y : It is understood that the PRODUCT of $Y^2 Z^2 = 1$, so $Z = \frac{1}{Y} = \frac{1}{\sqrt{b^2 + g^2}}$ and $Y = \frac{1}{z} = \frac{1}{\sqrt{r^2 + x^2}}$

E = Electromotive Force
r = Resistance
x = Reactance
g = Conductance and
b = Susceptance

From the above:

- Y is defined as the resolving CURRENT into two COMPONENTS, and that it is in PHASE and quadrature with EMF, or the ENERGY CURRENT and the WATTLESS CURRENT, and
- Z is defined as the resolving EMF into two COMPONENTS, and that it is in PHASE and quadrature with CURRENT or the ENERGY EMF and the WATTLESS EMF

In other words, Admittance is the reciprocal of IMPENDANCE, thatis,
$\left(Y = \frac{1}{Z} = \frac{1}{r - ix}\right)$ **when we substitute for the COMPLEX CONJUGATE relation of AC**

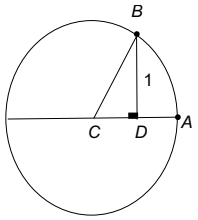
It is now necessary to descrie briefly, the notion of FORCE, also called the MAGNETOMOTIVE FORCE, and its application in INDUCTION MOTORS. For the relation T = mFϕSinα, where
T = Torque
m = Constant
F = Resultant MMF
ϕ = resultant Magnetism and
Sinα = Angle of hysteric advance of PHASE or the Magnetic Cycle
It is observed that the APPARENT EFFICIENCY of the motor is a FRACTION of the primary hysteric ENERGY and is < (1 – S) Sinα , where S represents SLIP or SYNCHRONISM.

- This is the base prerequisite knowledge, and sets a fine fundamental formalism for our purpose.

I strongly recommend an adequate comprehension of C.P.Steinmetz's excellent book titled, 'THE THEORY AND CALCULATION OF AC PHENOMENA' before you proceed, or you will likely and completely run out of runway with this text.

- An evaluation of the structure of *a* specified ALGORITHM, its relation to PATTERNS OF IRRATIONALS, INTEGER ARITHMETIC, and the METHODS of HIGH PRECISION MULTIPLICATION. This is Computer Science !

Let AC = 500 000 000 000, where AC = CB = Radius of A Circle



Then AD = AC – CD = AC – $\sqrt{AC^2 - 1}$ = 500 000 000 000 – $\sqrt{500\,000\,000\,000^2 - 1}$

Observe that CD < UNITY. Here we ' ll use the FORMULA AD = AC – Sqrt$\left[(AC)^2 - 1\right]$, to GENERATE a PATTERN OF IRRATIONALS called the CATALAN NUMBERS . The ALGORITHM for COMPUTING Catalan Numbers is described as a procedure of DIVISION with MULTIPLICATION SUBROUTINES. That is,
AD = $n\,T = \Theta\, n^{3 \log_2} + \frac{3\,n\,T}{2} = \Theta\, n^{1.5848625}$, where the EXPONENT of (n) is an IRRATIONAL NUMBER. Therefore,
we can conclude that for 4 multiplies of $\frac{n}{2}$ digit numbers taking $\left[\Theta\left(n^2\right)\right]$ time to compute, then
AD = $n\,T = \Theta\, n + \frac{4\,n\,T}{2}$ representing a LINEAR TIME for addition.
Its not so obvious at this point that an understanding of ALGORITHMS will be indispensable in the analysis of the FUNCTION $X_i = b\ m\ (\mathrm{mod}\ X_{i-1})$, noted as the CONGRUENTIAL MULTIPLICATIVE GENERATOR in Wolfram Language. (*There cannot be a passive treatment of RANDOM NUMBER GENERATORS*)

- Assuming we're now adequately prepared and equipped with the formal background to analyze and comprehend the STRUCTURE of VECTOR STATE SPACE, we'll learn that

REAL NUMBERS are represented as COMPLEX NUMBERS by showing in this instance if $\left(\frac{1}{\sqrt{2}}\right)$ is a COEFFICIENT or COMPONENT in a COMPLEX VECTOR, then its COMPLEX CONJUGATE is $\left(\frac{1-i}{2}\right)$ and that, the INNER PRODUCT or MAGNITUDE of the COMPONENT $\left(\frac{1-i}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$ such that $\frac{1}{\sqrt{2}} = \frac{1}{2}$ and thus is related as the square of the the length of the VECTOR.

- Representation of EIGEN VECTORS and EIGEN VALUES will be evaluated inclusive of the PAULI MATRICES $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}$, and $\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ without which,

the STRUCTURE and DEFINITION of STATE SPACE would break into oblivion.

- Finally, the OBJECTIVE will be to illustrate the TRUE and CLEAR meaning of the idea of ABSOLUTE SPACE, or the notion of CONSERVATION OF ENERGY .

This is only possible MATHEMATICALLY, as we shall see,
by applying the elegant FOURIER TRANSFORM PAIRS constructed from a fixed POISSON ' s COMMUTATOR RELATION and HEISENBERGS ' s representation of the matrix formulation of the way averages change with time.

$\int \partial \left(\frac{P}{2\pi}\right) e^{-iP(x-x')} = \delta(x - x')$

$\int \partial \left(\frac{x}{2\pi}\right) e^{ix(P-P')} = \delta(P - P')$ where ;

δ = Dirac delta function
P = Momentum
x = Position
c = Speed of light
$e^{iP(x-x')}$ = NORMALIZED or ORTHOGONAL BASIS VECTOR or HERMITIAN COMPLEX CONJUGATE

It follows then, that a CLEAR neat equation emerges from applying Schrodinger ' s WAVE EQUATION $\frac{\partial \psi}{\partial t} = -c \frac{\partial (x)}{\partial x}$ or the COMPLEX FUNCTION $\psi(x) = (x - ct)$ merged with the HAMILTONIAN description of the MOTION OF A PARTICLE in one DIMENSION moving along a line and in terms of a VECTOR STATE SPACE , yielding the conjugate relation If $\frac{\partial \mathcal{H}}{\partial P} = \dot{x} = c$, then $\frac{\partial \mathcal{H}}{\partial x} = -\dot{P} = 0$.

We ' ll conclude by defining ABSOLUTE SPACE as $\langle x \mid P \rangle \not< \frac{\hbar}{2}$

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1. SPRING-PROBLEM APPROXIMATION

relation between MASS and POTENTIAL

Consider the HAMILTONIAN description of the particle
moving under the influence of the Force exerted by the spring
$\mathcal{H} = \frac{P^2}{2m} + \frac{1}{2}\left(Kx^2\right)$ where
P = Momentum
m = Mass of the particle
K = Spring Constant and
x = Position of the particle
It is understood also that
$T = \frac{1}{2} mV^2$ and Classically,
p = mV
For a SMOOTH POTENTIAL,
we shall later in the text re - evaluate Momentum as described by the STATE VECTOR
$\dot{P} = F = -\frac{\partial V}{\partial x} = \frac{\partial P}{\partial t}$ which is the TIME DERIVATIVE of MOMENTUM and is equivalent to FORCE.
For now, let us specify the initial condition
as [x (0), P (0)] and there is no coupling to an external system.
If the Energy E is a CONSERVED QUANTITY,
the particle will follow a trajectory on a surface of CONSTANT ENERGY,
or SMOOTH POTENTIAL , or INERTIA, given by the relation
$\frac{Kx^2}{2E} + \frac{p^2}{2mE} = 1$. We can then describe the EQUATIONS OF MOTION
as the relation between VELOCITY and MOMENTUM by differentiation where
$\frac{\partial x}{\partial t} = \frac{\partial \mathcal{H}}{\partial P} = \frac{p}{m}$ is the MOTION OF *A* WAVE PACKET or the relation between VELOCITY and MOMENTUM,
while $\frac{\partial P}{\partial t} = -\left(\frac{\partial \mathcal{H}}{\partial x}\right) = -Kx$ implies that the TIME DERIVATIVE of MOMENTUM is FORCE.

- Suppose we would like to compute some PROPERTIES of the SYSTEM as it moves along a PATH

[x (t), P (t)] in PHASE SPACE using NUMERICAL
INTEGRATION. (*We shall later establish the equations of
motion in vector state space for an analytical treatment*)

We go about the solution of APPROXIMATING the CONTINUOUS PATH by a FIRST ORDER DIFFENTIAL taken as

$\frac{\partial f}{\partial t} = \frac{1}{h}\left(f(h+t) - f(t)\right)$, (*Refer to Euler's Method of Linear Approximation*)

(h) being the BASIC TIME STEP. Here the solution of the equation of motion is obtained at times which are multiples of the basic unit (h).

It follows that a RECURSION FORMULAE for the position and momentum can be obtained by inserting the differential into the equation of motion as

$\frac{\partial x}{\partial t} \simeq \frac{1}{h}(x(h+t) - t\,x) = \frac{p(t)}{m}$ and

$\frac{\partial P}{\partial t} \simeq \frac{1}{h}$ (p (h + t) − tP) = −Kx (t) or simply reshuffled as

$x(h+t) = \frac{hP(t)}{m} + (t)\,x$ and

$P(h+t) = P\,t - Khx(t)$.

So given an initial position $x(0)$ and momentum $P(0)$ consistent with a given Energy, the TRAJECTORY of the particle is SIMULATED at t = 2 h, 3 h, ..., etc noting that the TOTAL ENERGY is set to UNITY.

- So, the trajectories in PHASE SPACE for the SPRING PROBLEM is thus GENERATED with a simple ALGORITHM where if the path is spiral indicates that the Energy is absorbed by the particle. A more reasonable trajectory is the result indicating an almost CONSERVED ENERGY.
- This method simulates a simple and elegant approach to the spring problem. We are therefore interested in developing ALGORITHMS OF HIGH ORDER in (h) to keep the ERROR as small as possible, and a TIME-STEP that the SYSTEM reaches an EQUILIBRIUM during a run.

Suppose you were to Develop an algorithm and write a computer program for the pendulum

$\frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{g}{l}\right)$ Sin ϕ where (l) is the length of the string and (g) is the gravitational constant.

This is a SECOND ORDER differential equation, and the solution involves PARTIAL DERIVATIVES. We shall look into the method of INTEGRATION BY PARTS at a much later context.

ORGANIZATION OF SIMULATION

No matter how high the ORDER OF AN ALGORITHM, a system will eventually depart from the trace trajectory, as long as (h) is FINITE.

A drift in the energy δE is caused by the finite time step. This raises the question for the conservation properties of the algorithms during the simulation. To establish conservation is to CONSTRAIN the system the system artificially;
Instead of using Forces to calculate the motion we use POTENTIALS.In addition, we may ask for the TIME REVERSAL PROPERTIES noting only the one step method is INVARIANT under time reversal if we require that the equations define a TRANSFORMATION.

What may be the reason for Energy fluctuations? Why are the trajectories then NOT time-reversible?

To constrain the Energy, a FEEDBACK LOOP is established which introduces a TIME DELAY from the differential equations leading to fluctuations in the AVERAGE ENERGY(less significant though).

It is desirable to have an algorithm which changes a value from step to step. Initially, the values should be large to compensate negative results, and gradually decreases as the system approaches equilibration.What would be a computational scheme for the TRUNCATED OCTAHEDRON boundary condition?

DEMONS

Let us now develop an algorithm to specify the TRANSITION PROBABILITY from one state (x) of the system to a state (x') per unit time. The goal is to ensure positivity of the domains's Energy.

First, we note the transition probability as [W (x, x')]. To ensure that the states are ultimately DISTRIBUTED according to [P (x)] equilibrium states, restrictions must be placed on [W (x, x')].
For all complementary pairs $(s, \bar{s})$ of sets of PHASE POINTS,
there exists (x ∃ S) and $(x \exists \bar{s})$ such that [W (x, x') ≠ 0]. This means, for all (x, x'), [w (x, x') ≥ 0];
for all (x) the $\sum_{x'} W(x, x') = 1$; and
for all (x) the $\sum_{x'} w(x, x') P(x) = P(x)$.

The DISTRIBUTION of STATES must corespond to the EQUILIBRIUM DISTRIBUTION $\left[Z^{-1} f[H, \{x\}]\right]$ because [P (x)] = $\left[Z^{-1} f[H, \{x\}]\right]$ where (Z) is the proportionality constant and does not enter transition probabilities noting also its not accessible in a simulation.
This implies that the free ENERGY [F = $-K_B$ Tln (Z)] cannot be computed directly. We must then compute properties of the system using the proportionality constant (Z) since the KINETIC ENERGY term from the HAMILTONIAN is dropped and thus we cannot use the equations of motion.
Now, suppose that a state (x) is generated such that [H (x)] E representing a RANDOM WALK ON A CONSTANT SURFACE IN PHASE SPACE. A sampling algorithm on the surface then has to produce further states and allow for (∃) variations or equilibrium in the region [E − ∃ < H (x) < E + ∃] away from the surface. We do this by introducing a DEMON, known as a EXTRA DEGREE OF FREEDOM with Energy (E_D) into the proportionality constant such that $Z = \sum_x \sum_{E_D} \delta[H(x) + E_D - E]$ where the distribution is

expressed by a DELTA FUNCTION. So the AVERAGE VALUE of Z = $\int_\alpha \delta[H(x) - E]\, \partial x$.

You must know that With any OBSERVABLE (A) is associated *a* FUNCTION [A (x)] which is the MEAN AVERAGE. (* Refer to Probability Density Function *). Thus, for

$$A = \frac{1}{2} \int_\alpha A(x)\, \delta[H(x) - E]\, \partial x$$

implies sampling the available PHASE SPACE
of the system [α] and carrying out *a* SUMMATION of the function (Z).
The DEMON plays *a* role similar to the Kinetic Energy by producing changes in the configuration by traveling around the system and transferring Energy. It thus creates *a* Random Walk of the system on the surface. However,
we MUST restrict the Demon's by limiting its Energy to positive values otherwise it will absorb all the energy (compare with step 7 in the algorithm below).

Algorithmically, an outlined and summary of the procedure takes the form:

1. Construct a state such that [H (x) = E]
2. Set the Demon Energy (E_D) for example [(E_D) = 0]
3. Choose a part of the system
4. Change the local state of the system so that (x) truncates to (x')
5. Calculate the Energy change produced for instance [ΔH = H (x') − H (x)]
6. If the Energy is LOWERED, accept the change,
set $E_D \longleftarrow (E_D - \triangle H)$ and count (x') as a new configuration. Return to step 3.
7. Otherwise, accept the change ONLY IF the Demon carries enough Energy,
that is $(\triangle H - E_D) > 0$. In this case, $E_D \longleftarrow (E_D - \triangle H)$ and count (x') as a new configuration.
8. Return to step 3

The algorithm guarantees with step 6 and step 7 that the system relaxes to thermal equilibrium. In addition, step 7 also ensures the positivity of the DOMAIN'S energy.

Conceptually, we may view the Demon as a thermometer. Indeed, the Demon can take up or

lose Energy as it is successively brought into contact with the parts of the system.

Initially, the Demon has an arbitrary distribution. The system thus acts as a reservoir and renormalizes the Demon. Ultimately, the energies become BOLTZMANN DISTRIBUTED, allowing the calculation of the temperatures as

$$P(E_D) \propto \exp\left(-\frac{E_D}{T K_B}\right)$$

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Below is a schematic representation of a random walk on
 a constant Energy surface is Phase Space using the Wolfram Language
(*SeedRandom[1234];data2d=RandomFunction[RandomWalkProcess[0.5],{0,10^4},2];
Graphics[Line[Transpose@data2d["States"]],AspectRatio→Automatic]*)
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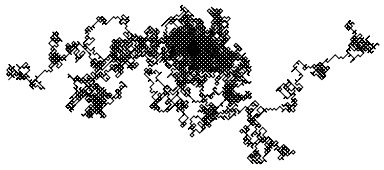

- We can see more clearly the meaning of the choice of transition of probabilities where the system is driven towards the minimum Energy. Thus, configurations which raise the energy are only accepted with a Boltzmann Probability.
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LOGICAL OPERATORS

The self-avoiding random walk is where the walker does not cross its own path. At each step, the walker checks if the neighborhood sites have been visited before. Of course, the walker is not allowed to retrace its steps. Quite often, the walker encounters a situation where all the sites in the immediate neighborhood have been visited before; The walk then terminates.

To be able to write a program which shows a screen how a self-avoiding random walk proceeds across a two-dimensional grid, one has to assume a RECURSIVELY SUMMING PSEUDORANDOM NUMBERS.

- Before we proceed further in the analysis of RANDOM NUMBER GENERATORS, get a clear understanding of the methods of LINEAR CONGUENTIAL GENERATORS and FEEDBACK SHIFT REGISTERS.
- Run the programs to get acquainted with the workings of the algorithms and to develop the requirements of a simulation.
- Find a thorough treatment on how to generate RANDOM NUMBERS in Donald Knuth's 'The Art Of Computer Programming Volume 2 (Chapter on Semi-numerical Algorithms). Hands-on experience with the C-Programming Language expected.

- Note that such an algorithm is an ITERATION of a fixed LINEAR TRANSFORMATION on a FIXED VECTOR SPACE.

By a general THEORY of LINEAR RECURRENCE, each entry in the ARRAY(VECTOR SPACE) is a LINEAR RECURRENCE SEQUENCE corresponding to the characteristic POLYNOMIAL.

For [(n*w) - r] array with a characteristic polynomial [ϕB (t)] of the TRANSFORMATION (B),
the sequence attains the maximal PERIOD $\left[2^P - 1\right] = 2^{(nw-r)} - 1$,
if and only if [ϕB (t)] is PRIMITVE (has ROOTS)
The test of PRIMALITY of an INTEGER is achieved by eliminating (r) BITS from
the (n*w) array so that the DIMENSION of the STATE SPACE is (nw - r) array. That is,
PRIMES of the form $\left(2^P - 1\right)$ upto (P = 1 398 269) need no FACTORIZATION.We ' ll later
understand the implication of COMPONENT FACTORIZATION and the concept of CORRELATION.
In determining the next STATE, each BIT must be fully REFLECTED thus results the recurrence,
and by MULTIPLYING with a MATRIX (T), called TEMPERING, a good K - Distribution is realized.
(*To fully appreciate the concept, get acquainted with the
Multiplicative Recursive Matrix Method and Inverse Decimation Method*)
Its interesting to note that in the case of (P) being very large (> 10 000),
the direct computation may need several years to catch a primitive
polynomial. A concise illustration is in the application of TIME COMPLEXITY.

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2. AN INTRODUCTION TO CRYPTOGRAPHY

For IT Security, its relevant to understand the classification, basic setups, SUBSTITUTION CIPHERS and attacks on crypto-systems. Some of the modern applications of Cryptography include:

- Cell Phones (GSM Voice Encryption)
- Bank Cards and Online Banking
- Electronic Passports
- Secure Shell, E-mail Encryption and Plug-ins for Web Browsers

These applications entail a two diverse unified subjects of CRYPTOLOGY. That is:

- Cryptography; Involves Symmetric, Asymmetric Algorithms(PKI) , and Protocols
- Crypto-analysis

The setup for symmetric cryptography raises the problems of communication over an insecure channel. Here the channel examples are the INTERNET, and WI-FI.

Note that Public Encryption Functions are the only ways so far, of keeping these channels SECURE(defined as CRYPTOANALYSIS).

In practice, one should never use an untested CRYPTO-ALGORITHM. In principle, a CRYPTOSYSTEM should be secure even if the attacker knows all the details about the system, with the exception of the SECRET KEY. Observe that the principle is counterintuitive.

The SUBSTITUTION CIPHER is a historical cipher that operated on letters of the alphabet. For example, if

A represents m

B represents d

C represents w

E represents q and so forth....We obtain a fixed look-up table. For instance, e[CBBC] represents (wddw). Is this cipher secure? How can it be broken?

1.

2^{88} divided into $(2^{56} * 2^{32})$ resulting in ≃
10^9 days TIME COMPLEXITY. This implies the keyspace it too large.
2. Letter Frequency Analysis where the letters have very
different frequencies in the English Language. Moreover,
the frequency of PLAIN TEXT letters is preserved in the CIPHER TEXT.
For instance, the letter (e) is the most common letter in English. Almost 13 %
of all letters in a typical English text are (e). The next most common letter is (t),
with about 9 %. This method of analysis works because identical plain texts map to identical
cipher text symbols thus Its weakness.So if an attack is succesful, then the system collapses.

There are often many possible attack approaches used to describe cryptosystem attacks. These are:

1. Classical Crypto-analysis; Includes the briefly described Brute-force and Analytical attacks(letter frequency analysis)

2. Social Engineering

3. Implementation attacks, where the HARDWARE is not protected. This is an interesting "side Channel analysis" of the key SPACE, where one records the power TRACE (walk) on an oscilloscope once the algorithm on the microprocessor has been triggered. The timing behavior (step) depends on the key BITS (array). We then use the timing code to compute the key.

DEFINITION :
A PSEUDORANDOM SEQUENCE (x_i) of (w_bit) integers of Period (P),
satisfying the following conditions is said to be K – Distributed
(*Refer to our discussion of our discussion on TEMPERING*)to (v_bit) accuracy :
Let $[\text{Trunc}_v(x)]$ denote the number formed by the leading
(v_bits) of (x_i) and consider (P) of the (Kv_bit) vectors such that
$[\{\text{Trunc}_v(x_i), \text{Trunc}_v(x_{i+1}), \ldots.., \text{Trunc}_v(x_{i+(K-1)})\}]$ for $[0 \leq i < P]$
where $[(i = 1, 1, \ldots.P - 1)]$ represents coordinates and
(K) represents the DIMENSIONAL UNIT
The SEQUENCE is said to be K – Distributed to (v_bit) accuracy if each DIMENSION contains
the same number of points $(\text{except for the origin})$. Consequently, the higher (Kv_bit)
for each (v_bit) assures a higher dimensional EQUIDISTRIBUTION with (v_bit) PRECISION,
based on a LINEAR RECURSION over the two ELEMENT FIELD (F_2),
known as the (F_2) GENERATORS.(*This should be apparent to you,
considering that a linear sequence has a characteristc polynomial*)

The cryptographic interpretation of the K-Distribution will assure that such a sequence is K-Distributed to (v_bit) accuracy, and that all the SEED(INITIAL) BITS are RANDOMLY GIVEN. Thus, the knowledge of the most significant (v_bits) of the first (w) words does not allow the user to make any statement about the most significant (v_bits) of the next word, if (w < K).This is because every BIT PATTERN occurs equally likely in the (v_bits) of the next word, of course, by definition of K-Distribution.

Thus, if the simulated system is sensitive only to the history of (K) or less previously generated words with (v_bit) accuracy, then it is THEORICALLY SAFE. (Refer to a detailed justification of the recurrence in the paper "MERSENNE TWISTER: a 623-Dimensionally Equidistributed PseudoRandom Number Generator" by Makoto Matsumoto and Takuji Nishimura)

Another and important criterion criterion on the RANDOMNESS of the (F_2) GENERATORS is in the number of terms in the characteristic polynomial of the STATE TRANSITION FUNCTION (*characteristic polynomial with many terms*)

We'll later compare the speed (CPU TIME) and WORKING AREA of pseudorandom number generators; Although high speed is probably due to the hardware architecture like CACHE MEMORY and PIPELINE PROCESSING.

- Note that the calculation of the recurrence is realized with BITSHIFT.
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3. PATTERNS OF IRRATIONALS, AND THE COMPLEXITY OF AN ALGORITHM

Its understood that the ratio of the diagonal on the square is INCOMMENSURABLE.

An output of the $\left(\sqrt{2}\right)$ in Wolfram Language, N[Sqrt[2], 18] outputs 1.41421356237309505

We now can define a Catalan Number as the
Set (P) of balanced parenthesis strings where [λ ∃ P] where λ is the empty string. Thus,
If [(α, β) ∃ P], then [(α) β ∃ P]
We can obtain a balanced parenthesis string via the second rule, from a unique (α, β) pair. Therefore,
an ENUMERATION of the CARDINALITY OF THE SET can be constructed as follows :
Let (C_n) be the number of balanced paranthesis strings with exactly (n) pairs of paranthesis,
where $C_0 = 1$ and is the empty string. Then $C_1 = 1 \Longrightarrow (\ )$,
$C_2 = C_0\, C_1 + C_1\, C_0$ where we choose the α, then β ; Thus $C_2 = 2$; and

$C_n = \sum_{k=0}^{n} C_k\, C_{n-k}$ $n \geqslant 0$ representing the GENERATOR.

Consider the Newton ' s Method of approximating SQUARE ROOTS

where $\{y = f(x)\}$. Then the derivative $\left\{y' = \frac{\partial y}{\partial x}\right\}$ = SLOPE OF THE TANGENT LINE

We find the roots of $f(0) = 1$ through succesive approximation. For example,
we take the equation $[y = f(x)]$ and note the TANGENT to this curve. Thus,
the equation of the TANGENT is the DERIVATIVE and so,
$x_0 = 1.000000000$; where we start with one digit PRECISION
$x_1 = 1.500\,000\,000$
$x_2 = 1.416\,666\,666 \quad \leftrightarrow 2$
$x_3 = 1.41215686 \quad \leftrightarrow 4$
$x_4 = 1.41\,421\,3562 \quad \leftrightarrow 8$
The important observations in this recurrence is that,
Integers with certain digits have a QUADRATIC CONVERGENCE where the number of digits double
with every ITERATION resulting in a LOGARITHMIC number of iterations. For instance,
note that (2, 4, 8, 16, ..., ...) is a GEOMETRIC SERIES.
If we take (a) to be an integer or fraction or a small number, and (X_i) to be a number with large digits,
we need COMPUTERS to divide (100 _ digit numbers) by (50 _ digit numbers).So,
for $\left[\left(\frac{a}{x_i}\right) * \left(\frac{1}{x_i}\right)\right]$ is referrred to as a HIGH PRECISION MULTIPLICATION
because it is a division algorithm with multiplication subroutines.
If we assume $\left(\sqrt{2}\right)$ to be a double digit precision as from above,
then we want an integer that is $\left\{\sqrt{2}\ 10^d\right\} \cong \left\{\sqrt{2 \times 10^{2\,d}}\right\}$ (*The symbol '≅' means Tilde_Full_Equal*)
This is the definition of the algorithm for applying
Newton ' s Method to form the desired high precision multiplication .
Note that (d) is the number of DIGIT PRECISION. For instance we may request for $(d = 10^6)$.

Let two (n_digit) with Radix (r = 2, 10) where the RANGE $[0 \leqslant (x, y) \leqslant r^n]$ is precise.

- We'll use the strategy of DIVIDE and CONQUER where we break the digits to 64_BIT, and let the machine do the rest of the computation.

Let $\{(x = x_1\, r^{n/2} + x_0)\}$; and
$\{(y = y_1\, r^{n/2} + y_0)\}$ where (x_1) is the high half and (y_1)
is the lower half. (* Obserser the characteristic polynomial*) Then,
$\{(0 \leqslant (x_0\, ,\, x_1) \leqslant r^{n/2})\}$ and $\{(0 \leq (y_0,\, y_1) \leqslant r^{n/2})\}$ thus a straighforward
decomposition of the multiplication operation;
let $\{(z_0 = (x_0\, y_0)\}$,
$\{(z_1 = (x_0\, y_1 + x_1\, y_0)\}$ and thus,
$\{(z_2 = x_2\, y_2)\}$ (*Refer to the enumeration of cardinality of the set,
at the beginning of the chapter*) Then, $\{(z = (xy) = x_2\, y_2\, r^n + (x_0\, y_1 + x_1\, y_0)\, r^{n/2} + x_0\, y_0)\}$
(*We've applied and substituted the characteristic polynomial
function to obtain the formula of the algorithm*)
Observe now that the (z) has 4 multiplies of $(n/2)$ digit numbers at the second term $\left[(x_0\, y_1 + x_1\, y_0)\, r^{n/2}\right]$;
which is equivalent to $\Theta\left(n^2\right)$ TIME to compute (* The $\log_2 4 = 2$ *).So
we can assuredly remark that (z) is a $\Theta\left(n^2\right)$ ALGORITHM.
Then in GENERAL, $T(n) = 4\, T(n/2) + \Theta(n)$ represents a LINEAR TIME for ADDITION,
calle d the COMPLEXITY OF THE ALGORITHM.
We can reduce the complexity further using an algorithm with 3 multications instead of as of $\Theta\left(n^2\right)$. So,
If $T(n)$ $3\, T(n/2) + \Theta(n) = \Theta\left(n^{\log_2 3}\right) = \Theta\left(n^{1.5848625}\right)$,
meaning the complexity of the algorithm is an IRRATIONAL NUMBER.

At the beginning of our Introduction, we supposed a really big circle with center (C) and a Diameter of 1 Trillion. We then dropped a perpendicular one UNIT high so that

CB = CA = r thus AD = AC – CD = $500\,000\,000\,000 - \left(\sqrt{500\,000\,000\,000^2 - 1}\right)$.
CD is not a perfect square and so is an irrational quantity. So by sutracting 1
from it we obtain an irrational quantity that goes to infinity. The algorithm for
AD = AC – CD computes the CATALAN NUMBERS.

■

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4. COMPUTATION COMPLEXITY

The MERSENNE TWISTER operates with (P = 19 937) which is a PRIMITIVE twisted generalized FEEDBACK SHIFT REGISTER SEQUENCE (*Review self-avoiding random walk*)

The algorithm does not involve any arithmetic since all operations are SHIFTS, AND's,OR's,XOR's.

A SEED(single integer) initiates the whole process generated upon start up with an unsigned 32_bit random integers.

There is TEMPERING with logical operations to improve RANDOMNESS. So, when the pointer reaches the end of the cache where the last element of the STATE is a pointer in cache, its refilled with another 623 ELEMENTS.

Our purpose is to analyze the group theoretic properties of PERMUTATION and TEMPERING operations.

First, we note that the PERIOD of the Mersenne Prime is $(2^{19\,937}-1)$. Its been demonstrated that the design generates results that satisfy an EQUIDISTRIBUTION property in 623_Dimensional cube. (Review the C_code for MT19937)

The code works with both 32_bit and 64_bit machines.

The function [genrand ()] returns a UNIFORMLY DISTRIBUTED REAL PSEUDORANDOM NUMBER of TYPE DOUBLE with 32 _bit PRECISION in the closed interval [0, 1]. The function [sgenrand ()] sets the INITIAL VALUE to the ARRAY {mt[N]}.Before using [genrand ()], [sgenrand ()] must be called with a NON_ZERO unsigned long integer as SEED.

There are ‘’Magic Numbers” in a macro table for replacing FUNCTION CALLS to enhance RUN-TIME. The GENERATOR has MULTIPLIERS that attribute a LINEAR CONGRUENTIAL SEQUENCE with extremely LOW SERIAL CORRELATION over the entire PERIOD, where the ACCURACY of the RANDOM SEQUENCE should be the SAME IN ALL DIMENSIONS.

RANDOM NUMBERS are truncated say to $(\lg V_t \_bits)$ where (V_t) decreases with increasing (t) and the (t_DIMENSION) is the the source of the random number $\left(\sqrt{x_1^2 + \ldots. + x^2{}_t}\right)$ such that $\left(x_1 + \ldots. + a^{t-1} x_t\right) \cong 0$ (modulo m). (*A low dimension being the most damaging to randomness*) The most generalized REAL VALUED DISTRIBUTION can be expressed in terms of its DISTRIBUTION FUNCTION $f(x)$, which specifies the PROBABILTY that a random quantity (X) will not exceed (x); That is $f(x) = P(r)\ [X \leqslant x]$ always increases monotonically from 0 to 1. This means $f(x_1) \leqslant f(x_2)$ if $x_1 \leqslant x_2$ and where $f(-\infty) = 0$ and $f(+\infty) = 1$.

If $f(x)$ is continuous and strictly increasing so that $f(x_1) \leqslant f(x_2)$ when $x_1 < x_2$, it takes on all values between 0 and 1 and there is an INVERSE FUNCTION $f^{-1}(y)$ such that, for $0 < y < 1$, $y = f(x)$ if and only if $x = f^{-1}(y)$. We can thus compute a random quantity (X) with distribution $f(x)$ by setting $(X) = f^{-1}(u)$ where (u) is UNIFORM with the PROBABILITY that $(u) \leqslant F(x)$. It follows then that NUMERICAL ANALYSIS is considered as good method for evaluating $f^{-1}(u)$ to a desired accuracy.

Understand that any algorithm that employs a random number as input will give a random quantity with some distribution as output. The task is to find general methods for constructing the algorithm, given the distribution function of the output.

From a Theoretical point of view, we consider optimal ways to generate random variables with a given distribution in the sense that the method produces the desired result from MINIMUM possible number of random BITS. For instance, in the HARD_CORE_BIT method, we consider

a Boolean function f such that $f(x)$ and $g(x)$ are easily computed, although $f\left[g^{-1}(x)\right]$ is NOT. (*For large modulus generators, the primitivity check is a hard number theoritic task to ensure a reliable source of random numbers*)

This is the reason for having a linear congruential sequence with low serial correlation over the entire period.

In PROGRAMMING, the absolute difference between two expressions could be evaluated at randomly generated points to test for INEQUALITY OF THE EXPRESSION.

For instance, there is no evidence that the $\sqrt{x^2}$ and the $|x|$ are different for REAL VALUES; while there is evidence that the $\sqrt{x^2}$ and the $|x|$ differ for at least some COMPLEX VALUES.

The point is, there are features in Programming Languages that seem a bad fit for the language. These includes the brevity of the language caused in part by the excessive reuse of symbols and problems with OPERATOR PRECEDENCE. Worse than that, many symbols and keywords are “overloaded”; given different meanings when used in different contexts. These make the SCOPE of the RULES not intuitively clear.

```
For example, the C symbol "()" has multiple different meanings
 1. Enclose formal parameters in a function definition
2. make a function call
3. Provide expression precedence
4. Convert (cast) a value to a different type
5. Define a macro with arguments
6. Make a macro call with arguments
7. Enclose the operand of "size_of" operator when it is type name
So, What does, "apple=size_of(int)*p"
 mean? Is it the size of an "int" multiplied by "p"? Or the size of whatever "p" points but cast
  to an int? Or something even weirder? Can one write a program to probe questions like this?
```

The more work you make one symbol or expression do, the harder it is for a compiler to detect anomalies in your use of it. This is an example of the C_Programming Language a little further out on the ragged edge of ambiguity! Further, it is not surprising that any change in PRECEDENCE will impose an intolerable burden on the existing source base and importantly the order of operand evaluation.

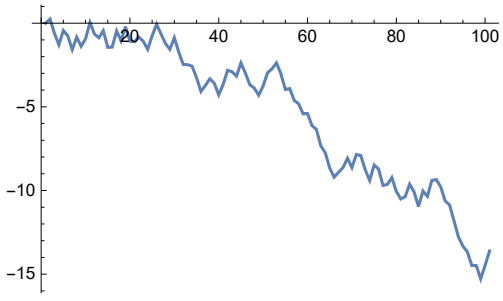
The solution is to always put parenthesis around an expression that mixes Booleans, arithmetic,or BIT_TWIDDLING with anything else. Besides, why is the order in which these groupings are evaluated undefined?

We'll now illustrate excellent and explicit application of parenthesis around expressions.

Below is a "random walk" generated in MATHEMATICA by recursively summing pseudorandom numbers, starting at 0;

```
ListLinePlot[Join[{0.}, Accumulate[RandomReal[{-1, 1}, {100}]]]]
```

Out =



Test for the equivalence of symbolic expressions by substitution of random numbers;

```
Max[Abs[Sqrt[x^2] - Abs[x] /. x → RandomReal[{-10, 10}, 10000]]]
Out = 0. (* This provides NO evidence that
 the √x² and the|x| are different for REAL VALUES *)

Max[Abs[Sqrt[x^2] - Abs[x] /. x → RandomComplex[{-10 + 10 I, 10 + 10 I}, 10000]]]
Out = 14.142(* This provides evidence that the
 √x² and the |x| differ for at least some COMPLEX VALUES *)
```

DEFINING RANDOM NUMBER GENERATORS IN WOLFRAM LANGUAGE :Writing the Code

```
MULTITIPLICATIVE CONGRUENTIAL GENERATOR
(*A multiplicative congruential generator follows the RECURRENCE RELATION [xi=b(xi-1)mod m]*)
(*The generator defined below, will alow ONLY for generation of REAL NUMBERS*)
(*Initialization of the generator will extract the values of the multiplier and
 modulus.Initialization will fail if either of these values is not a positive integer*)
(*Calls from the kernel to Random InitializeGenerator are effectively wrapped in
 Catch.Throw is used in the initialization code to easily exit in case of problems*)
(*"True" Establishes that MultiplicativeCongruential generates REALS*)
(*The following initializes the generator*)

Options[MultiplicativeCongruential] = {"Multiplier" → 123456789, "Modulus" → 2^35 - 1};
MultiplicativeCongruential /: Random`InitializeGenerator[MultiplicativeCongruential, opts___] :=
  Module[{mult, mod, flops = Flatten[{opts, Options[MultiplicativeCongruential]}]},
   mult = "Multiplier" /. flops;
   If[! (IntegerQ[mult] && Positive[mult]),
    Throw[$Failed]
   ];
   mod = "Modulus" /. flops;
   If[! (IntegerQ[mod] && Positive[mult]),
    Throw[$Failed]
   ];
   MultiplicativeCongruential[mult, mod, 1]];
MultiplicativeCongruential[___]["GeneratesRealsQ"] := True;
MultiplicativeCongruential[mult_, mod_, ___]["SeedGenerator"[seed_]] :=
 MultiplicativeCongruential[mult, mod, Mod[(mult*seed), mod]
MultiplicativeCongruential[mult_, mod_, s_]["GenerateReals"[n_, {a_, b_}, prec_]] :=
     Module[{x = s},
       {a + (b - a) Table[x = mult*x;
            Mod[x, mod], {n}] / mod, MultiplicativeCongruential[mult, mod, x]}
      ]
BlockRandom[
       SeedRandom[Method -> MultiplicativeCongruential];
       RandomReal[{15, 55}, 5]]
       (*Generates 5 reals using the MultiplicativeCongruential generator*)
Out : {35.19237603468915, 42.523274676278334, 49.298336948102175,
         42.65275420468687, 16.116889654691242}
```

```
BlockRandom[
 SeedRandom[Method -> MultiplicativeCongruential];
 RandomReal[{15, 55}, 5]]; // Timing
Out : {0., Null}
(* Add "//Timing" to see that the
 built in methods take almost no TIME TO COMPUTE the values *)

BlockRandom[
 SeedRandom[Method → MultiplicativeCongruential];
 RandomInteger[{5, 50}]]
Out :
... RandomInteger: "The current random generator does not support generation of random
    integers in the given range."
RandomInteger[{5, 50}]
(* It means the generator is not defined for integers *)
```

A Random Distribution Vector can also be used to define generators for multidimensional distributions.

Lets suppose a random point from a UNIFORM DISTRIBUTION on the UNIT DISK where,
the set of REAL points {x, y} with $\sqrt{x^2 + y^2} \leq 1$
is desired. Such a RANDOM point can be constructed as follows :

```
(* Generate a RANDOM ANGLE θ uniformly distributed on [0,2π] *)
(* Generate a RANDOM VECTOR (u) uniformly distributed on [0,1] *)
(* Return {Sinθ √u ,Cosθ √u}. The returned ORDERED PAIRS can be MULTIPLIED
 by (r) to generate points uniformly distributed on a disk of radius (r) *)
```

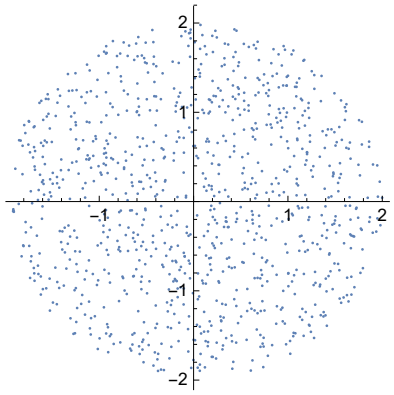
Below is the Wolfram Language Code :

```
(* "RandomReal" defines a generator for a uniform disk of radius (r) *)
(* "ListPlot" VISUALIZES the DISTRIBUTION of (10^4) generated points on this disk *)

 UniformDisk /:
RandomDistributionVector[UniformDisk[r_?Positive], n_Integer, prec_?Positive] :=
r * Sqrt[RandomReal[1, n, WorkingPrecision → prec]] *
 Transpose[{Cos[#], Sin[#]} & [RandomReal[2 Pi, n, WorkingPrecision → prec]]]

RandomReal[UniformDisk[2]]
Out : {-0.5549869088775746, 0.048327490357459944}

ListPlot[RandomReal[UniformDisk[2], 1000], AspectRatio → 1]
Out :
```



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5. VECTOR STATE SPACE

DINSTICTION OF THE STATE OF A SYSTEM

The AVERAGE VALUE of a VECTOR σ with an ANGLE of Cosθ for a q_bit is + 1 or − 1.
Note that the STATE is NOT a SET. This makes a VECTOR STATE a collection of MATHEMATICAL OBJECTS,
that is COMPLETELY ABSTRACT VECTORS. But first, keep in mind the notion of a COMPLEX CONJUGATE.
A COMPLEX NUMBER is a COMPLEX VECTOR SPACE.
FUNCTIONS can also form a VECTOR SPACE.
We shall RESERVE the terminology of a square matrix for it doesnt apply in this case.

A COMPLEX CONJUGATE VECTOR SPACE, is a SPACE of COMPLEX CONJUGATE VECTORS.

Take the symbol "<>" to represent ANGLULAR BRA_KETS.
The Rules of Arithmetic Operations shall apply in all cases henceforth .
Let $\mathcal{Z}\,|\,a\rangle \leftrightarrow \langle a\,|\,\mathcal{Z}^*$
If $|\,a\rangle$ implies $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$, then $\langle a\,|$ implies (α_1^*, α_2^*)
(* Laid in a ROW to reprsent a complex conjugate vector space *)
The INNER PRODUCT or DOT PRODUCT $|\,a\rangle\,\langle b\,|\,= \langle b\,|\,a\rangle$ and note that $\langle b\,|\,a\rangle \neq \langle a\,|\,b\rangle$
although they relate to the complex conjugates
So, $\langle b\,|\,a\rangle = \langle b\,|\,a\rangle^*$;
$(\beta_1^*\,\beta_2^*)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \beta_1^*\,\alpha_1 + \beta^*_2\,\alpha_2$ (* This is the sum of the product the COMPONENTS*) and
$(\alpha^*_1\,\alpha^*_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \alpha^*_1\,\beta_1 + \alpha^*_2\,\beta_2$ (* The complex conjugate of the above *)
This is an illustration of a complex vector space.
The INNER PRODUCT of a VECTOR $\langle a\,|\,a'\rangle = \langle a'\,|\,a\rangle$ = REAL and + ve. Thus,
$(\alpha^*_1\,\alpha^*_2)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \alpha^*_1\,\alpha_1 + \alpha^*_2\,\alpha_2$ = REAL and + ve.
This is equivalent to the SQUARE of the VECTOR.
It is obvious and not obsecure to say that complex vector spaces have REAL and + ve LENGTHS.
If the INNER PRODUCT $\langle b\,|\,a\rangle = 0$, It is DEFINED as being ORTHOGONAL or PERPENDICULAR .

The DIMENSION OF A VECTOR SPACE is the maximum number of ORTHOGONAL VECTORS you can find in the vector space. Also referred to as MUTUAL ORTHOGONAL VECTORS. The reverse of the statement is TRUE.

For instance, $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$ are ORTHOGONAL VECTORS because 1 (0) + 0 (1) = 0.

This is a 2 DIMENSIONAL VECTOR SPACE and there cannot be a THIRD DIMENSION because of MUTUAL COMPONENTS.

REAL NUMBERS are a special case of COMPLEX NUMBERS and that, $\begin{pmatrix}0\\0\end{pmatrix}$ is ORTHOGONAL TO EVERYTHING.

The OUTER PRODUCT is represented as $|a><b|$ (* Observe the NOTATION *)

It is important to note that COMPLEX FUCTIONS form COMPLEX VECTOR

SPACES where the VARIABLES are COMPLETELY UNCORRELATED. (* RANDOMNESS *)

The AVERAGE VALUE of $<\sigma_m>$ = n * m

which implies the Cosine of the angle between (m) and (n) or the COMPONENT of (n) along (m).

The RULES OF CALCULUS OF PROPOSITION LOGIC (AND) & (INCLUSIVE OR) apply. Let,

A : $\sigma_{\mathcal{Z}}$ = +1

B : σ_X = +1

while $\mathcal{Z}$ = +1 ,

A or B is TRUE, B orA is TRUE

Then,

$<A|B>$ = COMPLEX VECTOR = COMPLEX CONJUGATE OF $<B|A>$. We also know that $|A>+|B>$ is a KET VECTOR;

So $\mathcal{Z}\,|A>$ is KET VECTOR multiplied by a COMPLEX NUMBER.

Recall that $<A|A>$ = REAL ,

which is analogous to the SQUARE OF THE LENGTH OF A VECTOR

(* That is, the INNER PRODUCT OF VECTOR and ITSELF *)

$<A|B>$ = 0 is an ORTHOGONAL (MUTUAL) VECTOR SPACE which is now an UNAMBIGOUSLY DISTINGUISHABLE

(* That is, the INNER PRODUCT OF TWO DIFFERENT VECTORS *)

These are the RULES for distinguishing a VECTOR SPACE.

Lets now examine the PROBABILITIES OF VECTOR SPACES. Disregard whether the coordinate system is right or left handed.

Note that $\{-i*i=1\}$ and the $\sqrt{-1}$ = i;

So $\{e^{i\theta}\}$ will be considered as a PHASE FACTOR or UNIT LENGHT

Consider the following VECTOR STATES in a coordinate system where

$|R>$

$|L>$

$|I>$ and

$|O>$ represent RIGHT, LEFT, IN, and OUT while

$|u>$ and

$|d>$ represent UP and DOWN

For the KET VECTORS is a state of $|u>$ and $|d>$,

$<u|d>$ = 0

Then the PROBABILITY (sum of square of the length) for

$$|R> = \frac{1}{\sqrt{2}}\,(|u>+|d>)$$

$$|L> = \frac{1}{\sqrt{2}}\,(|u>-|d>)$$

(* The probabilities are the MAGNITUDES

of the VECTORS or the square of the length of a vector *)

Thus $\langle R \mid L \rangle = 0$ which is analogous to complex conjugation

$$\mid I \rangle = \frac{1}{\sqrt{2}} \mid u \rangle + \frac{i}{\sqrt{2}} \mid d \rangle \quad \text{and}$$

$$\mid O \rangle = \frac{1}{\sqrt{2}} \mid u \rangle - \frac{i}{\sqrt{2}} \mid d \rangle$$

We then consider COMPONENTS for multiplying vector spaces

If $\mid u \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mid d \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

Take $\mid U \rangle = \mid L \rangle + \mid R \rangle$ and $\mid d \rangle = \mid L \rangle - \mid R \rangle$; Then from above,

$$\mid R \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mid L \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \mid I \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \text{ and } \mid O \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$$

You can easily prove that $\mid I \rangle$ and $\mid O \rangle$

are ORTHOGONAL since the rule for calculating inner products requires that $\langle I \mid O \rangle = 0$

and also show that the row vectors have complex conjugates ,

meaning the SQUARE OF THE MAGNITUDE OF THE COEFFICIENTS MUST $= \frac{1}{2}$.

Thus, in the $\mid I \rangle$ STATE, the probability of $\mid u \rangle$ or $\mid d \rangle$ is most strikingly $\frac{1}{2}$. Here is the PROOF;

Lets us illustrate $\mid I \rangle$ in terms of $\mid L \rangle$ and $\mid R \rangle$. For

$$\mid I \rangle = \frac{1}{\sqrt{2}} (\mid L \rangle + \mid R \rangle) = \frac{1}{2} (\mid L \rangle + \mid R \rangle) = \frac{1}{2} \mid L \rangle + \frac{1}{2} \mid R \rangle + \frac{i}{\sqrt{2}} \left(\frac{\mid R \rangle}{\sqrt{2}} - \frac{\mid L \rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{2} \mid L \rangle + \frac{1}{2} \mid R \rangle + \frac{i}{2} \mid R \rangle - \frac{i}{2} \mid L \rangle$$

$$= \frac{1 - i}{2} \mid L \rangle + \frac{1 + i}{2} \mid R \rangle$$

Its important to VERIFY that the

complex number $\frac{1 - i}{2}$ multiplied by its complex conjugate MUST $= \frac{1}{2}$. Thus,

$$\frac{(1 - i)(1 + i)}{4} = \frac{(1 - i + i + 1)}{4} = \frac{2}{4} = \frac{1}{2}.$$

The statement above applies to ALL the PROBABILITIES of STATES of other combinations. The states are SYMMETRICALLY RELATED to each other. Its an easy concept but very unfamiliar indeed.

2D SPACE can now, clearly be represented as 2D VECTOR SPACE. A thorough understanding of TENSOR CALCULUS is required to proceed. Interestingly, it takes two REAL NUMBERS to pick a direction.

NOTE that when | d > and | L > are OVERLAPPING,
the PROJECTION PROPOSITIONS are NOT COMMUNITATIVE. (* We'll illustrate this further in the text *)
Moving forward,
If $|A\rangle = \sum_i \alpha_i |i\rangle$ then,

$\langle J|A\rangle = \sum_i \alpha_i \langle j|i\rangle$ and can be represented as δ_{ji}. This is the KRONECKER DELTA symbol.

So, the KET $|A\rangle = \sum_i |i\rangle\langle i|A\rangle$ where

$|A\rangle$ is written as the sum of the inner product of the BASIS or UNIT VECTOR and thus,

the BRA $\langle A| = \sum_i \langle A|i\rangle\langle i|$. This is the EXPRESSION of the VECTOR SPACE in terms of its COMPONENTS.

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6. LINEAR OPERATORS

An operator can be defined as a procedure (M) acting on $|A>$ to give $|B>$. That is $M|A> = |B>$.

So $M[\mathcal{Z}|A>] = \mathcal{Z} M|A>$ (* The relation is equivant *)

We ' re now intersted in the components of the output vector $|B>$.

We do this by projecting the inner product of both vectors using (i);

Thus for the case of of $M|A> = |B>$ we obtain $<i|M|A> = <i|B> = \beta_i$.

The $\sum_j <i|M|j><j|A> = <i|M|j>\alpha_j = \beta_i$. So $\sum_{ij} M_{ij}\alpha_j = \beta_i$. This can be illustrated

by multiplying *a* rule with *a* MATRIX by *a* column vector. For instance,

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & .. \\ M_{21} & M_{22} & M_{23} & .. \\ . & . & . & .. \\ . & . & . & \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_N \end{pmatrix}.$$

We can the define $<A|M|B>$ by noting $[<A|M]|B> = <A|[M|B>]$. So now,

the $\sum_{ij} <A|i><i|M|j><j|B> = \sum_{ij} \alpha^*{}_i M_{ij} \beta_j$.

If $M|A> = |B>$,

we note that $<A|M = <B|$ is COMPLEX CONJUGATE result (* Similar to $\alpha^*{}_i$ above *) Thus,

the correct NOTATION for this relation should be understood and written as

$<A|$

$M^\dagger = <B|$ where the symbol "†" is the Greek symbol for DAGGER and is called the HERMITIAN CONJUGATION.

Let us suppose that $M|i> = |i'>$ and $<j|M|i> = <j|i'>$. If this relatation is TRUE,

then $<i|M^\dagger|j> = <i'|j>$.

So the relation of two vectors $<A|$

$B>$ and $<B|A>$ is once again a COMPLEX CONJUGATION or TRANSPOSITION.

In our case the (i) and (j^{th}) matrix element of $M^\dagger$ is the complex

conjugate of the (j) and (i^{th}) matrix element of M. Thus,

we can rewrite $<i|M^\dagger|j> = <j|M|i>^*$ to indicate that there is an INTERCHANGE of (i) and (j).

A simpler relation is ${M^\dagger}_{ij} = M_{ij}{}^*$. This is the notation from the Mathematician HERMITE,

which is a REFLECTION about a DETERMINANT in the MATRIX

and then proceeding with complex conjugation. For example,

the Hermitian Conjugation of the matrix

$$\begin{pmatrix} 2 & 6+i \\ 4-i & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4-i \\ 6+i & 9+i \end{pmatrix}$$ (* First reflect about the diagonal *)

$$= \begin{pmatrix} 2 & 4+i \\ 6-i & 9+i \end{pmatrix}$$

(* Interchange the SIGN of the reflection and thus the complex conjugate as the final result *)

The Hermitian Conjugate so far turns a BRA VECTOR into a KET VECTOR; and in terms of MATRICES,

it flips (i) and (j); From the example above $(6+i)$ is TRANSPOSED to $(4-i)$,

then complex conjugated to $(4+i)$ and $(6-i)$ respectively.

The common notation $(\mathcal{Z} = \mathcal{Z}^*)$ representing a REAL NUMBER

is now replaced by HERMITIAN COMPLEX CONJUGATION OF MATRICES. So,

$M = M^\dagger$ and where the DIAGONAL ELEMENTS ARE ALWAYS REAL (4 and 6)

while the COMPLEX NUMBERS ARE COMPLEX CONJUGATED $(+i$ and $-1)$

EIGEN VECTORS

This can be defined as $M|I> = \lambda_I |I>$ where (M) represents HERMITIAN and (I) represents EIGEN VALUES.
If a matrix is Hermitian and the dimension of space is (n),
then they will be n_mutual ORTHOGONAL EIGEN VECTORS
(* This defines a BASIS UNIT or BASIS VECTOR *)
From the definition above,
the EIGEN VALUES of HERMITIAN MATRICES are then REAl. So $<I|M^\dagger = <I|\lambda_I^*$ means
λ_I is flipped and complex conjugated. So from the definition $M|I> = \lambda_I |I>$
$<I|M = <I|\lambda_I^*$.
We can take the INNER PRODUCTS of both the equations and obtain,
$<I|M|I> = \lambda_I <I|I>$ and
$<I|M^\dagger|I> = <I|$
$I> \lambda_I^*$. If we substitute the similirities of the two equations and cancel out $<I|I>$, then
$\lambda_I = \lambda_I^* =$ REAL.
So , If a matrix is Hermitian, their Eigen Values will be equal to complex conjugates and they are REAL.
Recall that we mentioned ORTHOGONALITY implies UNAMBIGOUSLY
DISTINGUISHABLE. Let us now PROVE that different EIGEN VECTORS with DIFFERENT
EIGEN VALUES are ORTHOGONAL, that is , they are NOT complex conjugates.
Let $M|I> = \lambda_I |I>$ and
$M|J> = \lambda_J |J>$. First, note
$<J|M = J|\lambda_J$ and so we have the two equations $M|I> = \lambda_I |I>$
$<J|M = J|\lambda_J$. We can the multiply both the left_hand
sides with (J) and (I) respectively to realize a substitution as we did earlier. So,
$<J|M|I> = \lambda_I <J|I>$
$<J|M^\dagger|I> = <J|I> \lambda_J$. So we have $\lambda_I = \lambda_J$;
The requirement of the PROOF is that $<J|I> = 0$. If we assume a value of 0 for the INNER PRODUCT,
we obtain a BASIS VECTOR or an ORTHORNORMAL VECTOR.

So far, we now have adequate MATHEMATICAL FORMALISM to INTERPRET the PHYSICAL MEANING OF THE STATE OF A SYSTEM.

We begin with the notion that OBSERVABLES are represented by HERMITIAN OPERATORS where the POSSIBLE VALUES ARE EIGEN VALUES of ($\pm$1) and while EIGEN VECTORS CORRESPOND to the EIGEN STATES(Meaning that they are unambiguously distinguishable)

The STATE of a SYSTEM is only OBSERVABLE when it is prepared or CONSTRUCTED using EIGEN VECTORS with EIGEN VALUES.

So ,
the PROBABILITY of λ_I,
is represented by the SQUARE OF THE INNER PRODUCT of the VECTOR | A > with (I). That is,
λ_I = < A | I > < I | A > (* The PROJECTION of |A> into the (I) DIRECTION *)

So For $\sum_I$ < A | I > < I | A > = 1 simply means that

< A | A > = 1 which is the sums of the square of the components which is = 1;

Thus the $\sum_I \alpha_I^* \alpha_I = 1$.

If we multiply the STATE VECTOR by a PHASE, for instance $\left(e^{i\theta}\right)$,
then ALL PROBABILITIES ARE INVARIANT or UNCHANGED.This implies FEW independent COMPLEX NUMBERS.

Let us now describe the ELEMENTS OF PAULI MATRICES which will be the
EIGEN MATRIX VALUES applied as the OPERATOR representing the COORDINATES :

$\sigma_z : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_x : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$\sigma_y : \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

So that an Eigen Matrix Value,
multiplied by the Eigen Vector should = A Basis or Eigen Vector (* Orthonormal Vector*) That is,
for σ_z : $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \{1*1\} + \{0*0\} \\ \{0*1\} + \{-1*0\} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. You can also show that σ_x and σ_y are Eigen Matrices.

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7. TIME EVOLUTION OF A SYSTEM
CONSERVATION OF ENERGY AND SPIN

We shall now analyze Mathematically, the notion of Linear Hermitian Observables.

The TIME_DEPENDENT SCHRODINGER EQUATION is represented as $i \frac{\partial \mid \psi >}{\partial t} = -i \mathcal{H} \mid \psi >$ while

The TIME_INDEPENDENT SCHRODINGER EQUATION is represented as $\mathcal{H} \mid E_i > = E_i \mid E_i >$

(* The summation convention does not apply for the repeated indices of E_i *) Then,

$\frac{\partial}{\partial t} < \psi_t \mid L \mid \psi_t > = -i < \psi \, (LH - HL) \, \psi >$ where

(L) represents AVERAGE VALUES

($\mathcal{H}$) represents Hamiltonian

(E) represents Eigen Vectors or Values

So $\frac{\partial}{\partial t} < L > = -i < [L, H] >$ where the shortened version is $\frac{\partial L}{\partial t} = -i [L, H]$

meaning that the COMMUTATOR [L, H] of a REAL is ALWAYS IMAGINARY.

We ' ll now introduce the POISSON BRAKETS and the COMMUTATOR.

The properties of Poisson Brakets is such that {A, B} = -{B, A}; So the COMMUTATOR [A, B] = -[B, A].

If we apply DIFFERENTIATION RULES for the PRODUCT {AB, C}, we obtain

{AB, C} = A {B, C} + {A, C} B so the COMMUTATOR [AB, C] = A[BC] + [AC] B and where

(AB, C - C, AB) = A (BC - CB) + (AC - CA) B

thus the COMMUTATOR satisfies the ALGEBRAIC RULE of the Poisson Brakets.

For the DIMENSIONAL STATES, $\{A, B\} = -\frac{i}{\hbar} [BA]$ can be rewritten as $\hbar \{A, B\} = -i [BA]$

where $\hbar$ is the PLANCK ' S CONSTANT. This implies the COMMUTATOR is negligibly small.

So TIME DEPENDENCE is realized as $\mid \psi_t > = \sum_j \alpha_{j(t)} \mid j >$ where α_j are the TIME_DEPENDENT COEFFICIENTS.

Thus, $\frac{\partial \psi}{\partial t} = \sum_j \frac{\partial \alpha_{j(t)}}{\partial t} \mid j >$

If we now insert the SCHRODINGER EQUATION to the equation above we write the DERIVATIVE as

$$\mid \dot{\psi}_{(t)} > = \sum_j \alpha_{j(t)} - i \mathcal{H} \mid j > = \sum_j \alpha_{j(t)} - i \mathcal{H} \mid E_j >$$

$$= -i \sum_j \alpha_{j(t)} \mathcal{H} \mid E_j >$$

We know that the SUM of two VECTORS MUST have EQUAL COEFFICIENTS. So,

$\frac{\partial \alpha_j}{\partial t} = -E_j \alpha_j$. This illustrates how the COEFFICIENTS change with TIME.

Therefore, the solution of the DIFFERENTIAL EQUATION of the COEFFICIENTS is an EXPONENT. That is,

$\alpha_j = \alpha_{j\,(0)} e^{-iE_j t}$. (* Recall that $\frac{\partial}{\partial x} = n \left[x^{n-1}\right]$ *)

Now that we know the solution for the coefficients (α),

we can then GENERALIZE the SCHRODINGER EQUATION at

$|\psi_t> = \sum_j \alpha_{j(0)} e^{-iE_j t} |j>$

where the coefficient (α) at time

(t_0) is the ENERGY DEPENDENT STATE $\left(e^{-iE_j t}\right)$ and is the PHASE.

The PHASE, is the RELATION between ENERGIES and OSCILLATIONS. That is, the WAVE FUNCTION.

So the sum of $\langle\psi_t|L|\psi_t\rangle = \sum_k \alpha^*_{k(0)} e^{iE_k t}\langle k|L|j\rangle \sum_j \alpha_{j(0)} e^{-iE_j t}$

Note now from the GENERALIZED SCHRODINGER EQUATION that

$\langle\psi_t| = \sum_j \alpha^*_{j(0)} e^{+iE_j t}\langle j|$

. This is represented by the

NOTATION (L

$\sum_{jk} \alpha^*_{k(0)} \alpha_{j(0)} L_{kj} e^{i(E_k - E_j)t}$ is the HEISENBERG's representation of the matrix formulation of the way AVERAGES CHANGE WITH TIME.

This means that the two FREQUENCIES E_k and E_j are associated with a MATRIX OBSERVABLE.

So (L) CHANGES WITH TWO FREQUENCIES IN TIME.

Note that the MAGNITUDES of the COEFFICIENTS stay FIXED
and the TIME EVOLUTION is associated ONLY with the RELATIVE PHASES.
We shall now evaluate the relation $\mathcal{H} = \frac{\omega}{2}\sigma_z$.
We can now apply the PAULI MATRICES mentioned to
determine the STATE OF PREPARED SYSTEM in three coordinates (x, y, z) where

$\dot{\sigma}_x = -i[\sigma_x\ \sigma_z]\ \frac{\omega}{2}$

$\dot{\sigma}_y = -i[\sigma_y\ \sigma_z]\ \frac{\omega}{2}$ and

$\dot{\sigma}_z = -i[\sigma_z\ \sigma_z]\ \frac{\omega}{2} = 0$ (* You can work out the Matrices to prove its ORTHOGONAL ;

Solution for working out matrices is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ ={{ar+bt,as+bu},{cr+dt,cs+du}}*)

The denominator (2) is arbitrary and will ease the substitution procedure later on.
Therefor, the COMMUTATION RELATION of the SIGMAS (σ) can be worked out as :

(a) $[\sigma_z\ \sigma_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 2\,i\,\sigma_y$

Here we have taken the product of the commutators of the matrices. Recall
to change the sign because of the order of the commutator. We thus get a +
ve sum in the second step. (* Carefully work through the manipulation *)
Recognize $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ being related to σ_y which is $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. If we multiply i by σ_y we obtain
$i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This NEATLY represents $2\,i\sigma_y = [\sigma_z\ \sigma_x]$. Therefore we can ascertain that

$[\sigma_z\ \sigma_x] = 2\,i\sigma_y$

$[\sigma_x\ \sigma_y] = 2\,i\sigma_z$ and

$[\sigma_y\ \sigma_z] = 2\,i\sigma_x$.

We can substitute the values of the commutators to obtain :

$\dot{\sigma}_x = -i\left[-2\,i\right]\ \sigma_y\ \frac{\omega}{2} = -\omega\sigma_y$

$\dot{\sigma}_y = -i\left[2\,i\sigma_x\right]\ \frac{\omega}{2} = \omega\sigma_x$

and

$\dot{\sigma}_z = 0$

Its important to note that (i) disappers since $(-i * -i = -1)$ and
$(-i * i = 1)$. This means that whem commutators have imaginary coefficients,
the TIME DERIVATIVES definitively have a REAL outcome in general.
Physically, This implies a PRECISION of $\dot{\sigma}_x$ and $\dot{\sigma}_y$ along the $\dot{\sigma}_z$ axis;
similar to a gyroscopic precision or a rotor in a magnetic field in CLassical Mechanics.

A 2D SPACE is the generalized HAMILTONIAN due to lack of independent operators.

Note that ω is the product of the magnetic field and magnetic moment of the spin.

We can thus conclude that the ENERGY level of the system $\mathcal{H} = \frac{\omega}{2}\sigma_z$ is ALWAYS $\pm\frac{\omega}{2}$.

If it is flipped, we get ω ENERGY. This means the SYSTEM is in a SUPERPOSITION STATE of either (± 1).

If its -1 there is NO RADIATION; while $+1$ there's RADIATION.

Thus, LINEAR OPERATORS act on EIGEN VECTORS and
result in ORTHOGONAL EIGEN VECTORS where the AVERAGE VALUES
must $\neq 0$ for a SINGLE VECTOR STATE.

For a COMPOSITE VECTOR STATE the system cannot be a seperate state and so the AVERAGE VALUE $= 0$.
This is a STATE of ENTANGLEMENT and is briefly outlined in the next chapter.

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8. STATISTICAL CORRELATION AND FUNCTIONS OF VECTOR STATES

What does it mean for an AVERAGE VALUE to be (-1) ? Here the situation is not of an inner product state.

The state of a composite vector is mostly oppositely oriented where the Eigen vectors are CORRELATED statistically. Do not think of this in a Classical system otherwise youwill encounter INCONSITENCIES. This property cannot be simulated. The correlation of a composite vector occurs when the coefficients do not FACTORIZE. Reflect back to the construction of the commutator of σ_z; So the product σ_z and τ_z = -1

In a PRODUCT STATE there is NO CORRELATION. We can thus know information about a COMPOSITE SYSTEM but NOT the CONSTITUENTS. Its VECTOR STATE allows this nonsensible notion because it totally Mathematical. The low Energy of a state implies correlation. Sit down and penetrate through this by summing over a vector multiplied by a matrix. The DENSITY MATRIX for a PRODUCT or PURE STATE is = 1.

At what point does the interpretation become un_Mathematical? Keep in mind the procedure is a DERIVATIVE of TIME.For instance, two connected computers cannot share information INSTANTENOUSLY. when you write a program to simulate a Vector State Space, accurate result simulation or the estimating precision is logically different. So dont be fooled. This means MEASUREMENTS are IRREVERSIBLE in UNITARY MATRICES. This is nature of simulators. Pretty intersting stuff so far.

FUNCTIONS OF VECTOR STATES

Let us now analyze OBSERVABLES, using PARTIAL DERIVATIVES.

Let us first define a FUNCTION where :

$D\,|\psi> \longrightarrow |\psi'>$ So we can then say that $D\,\psi(x) = \frac{\partial \psi}{\partial x}$. Is it HERMITIAN ?

The function can be written as an INTEGRAL . Observe.

Take the equation (a) to be $\int \phi^*(x)\,\frac{\partial \psi}{\partial x} = \int \psi^*(x)\,\frac{\partial \phi}{\partial x}\Big]^*$.

The right_hand side can be interpreted

as $\int \psi(x)\,\frac{\partial \phi^*}{\partial x}$ if it is Hermitian and is similar to the left hand side.

This is realized by INTEGRATION BY PARTS where the function or endpoints = 0.

This is done by by SHIFTING THE DERIVATIVE $\frac{\partial \psi}{\partial x}$ to

$\frac{\partial \phi^*}{\partial x}$ but we MUST now SHIFT THE SIGN of the left_hand side to get the result :

$-\int \phi^*(x)\,\frac{\partial \psi}{\partial x} = \int \psi(x)\,\frac{\partial \phi^*}{\partial x}$. Notice that the equation is NOT EQUAL.

There is a - ve ,

meaning the equation is opposite of each other.

We can fix the situation by applying a (-ve) Hermitian Operator such $\left(-i\,\frac{\partial}{\partial x}\right)$ defined as (P).

So from equation (a) with (P) inserted we obtain :

$(-i)\int \phi^*(x)\,\frac{\partial \psi}{\partial x} = (+i)\int \psi^*(x)\,\frac{\partial \phi}{\partial x}\Big]^*$

where (P) can represent MOMENTUM and written as $\left(\frac{P}{\hbar}\right)$. Here $\hbar$ is to give the relation a sensible UNIT.

Then the AVERAGE VALUE of $<\psi\,|\,L\,|\,\psi> =$

$\int \psi^*(x)\,\left(-i\,\frac{\partial}{\partial x}\right)\psi(x) =$ REAL. The proof of this relation is in the integration by parts.

This is illustrated by the shifting of the derivative and thus realizing a + ve (i) as shown below :

$-i\int \psi^*(x)\,\frac{\partial \psi}{\partial x}(x) = +i\int \frac{\partial \psi^*}{\partial x}\,\psi(x)$.

Thus, $<\psi\,|\,L\,|\,\psi> = \int \psi^*(x)\,\left(-i\,\frac{\partial}{\partial x}\right)\psi(x) = -i\int \psi^*(x)\,\frac{\partial \psi}{\partial x}(x) = +i\int \frac{\partial \psi^*}{\partial x}\,\psi(x)$.

Observe the complex conjugation

where ψ^* becomes $\frac{\partial \psi^*}{\partial x}$. You MUST CLEARLY see the complex conjugates.

TWO VECTORS that are COMPLEX CONJUGATES of each other are REAL.

We can now rewrite that :

$<\psi\,|\,L\,|\,\psi> = -i\int \psi^*(x)\,\frac{\partial \psi}{\partial x}(x) =$ REAL and thus is a HERMITIAN OPERATION and an OBSERVABLE.

The COMMUTATOR and the POISSON BRAKETS are related by the value $(i\hbar)$ where

$[x, P] = i\hbar$ and $\{x, P\} =$

1 where $(P) = -i\,\frac{\partial}{\partial x}$. Keep this relation in mind. It is the generalized Hermitian Operator.

For now, lets proceed with a Mathematical interlude.

FOURIER ANALYSIS

Most FUNCTIONS can be represented in the form of a WAVE. That is,
e^{iPx} as Cosines and Sines where (P) is OSCILLATION and is modulated either as a short or long wavelength.
Each wavelength has an AMPLITUDE which can be represented as $\tilde{\psi}(P)\, e^{iPx}$. So we can define the function
$\psi(x) = \int \frac{\partial P}{\sqrt{2\pi}} \tilde{\psi}(P)\, e^{iPx}$. Note that $\sqrt{2\pi}$ is *a* CONVENTION. So,

$\tilde{\psi}(P) = \int \frac{\partial x}{\sqrt{2\pi}} \psi(x)\, e^{-iPx}$. This is the BASIC FORM of the FOURIER THEORY where $\psi(x) = \tilde{\psi}(P)$,
thus $\tilde{\psi}(P)$ is the FOURIER TRANSFORM of $\psi(P)$ with a (-ve) sign.
This expresses the relation between (x) and (P) which is used to DEFINE POSITION and MOMENTUM.

Let us now DEFINE the properties of the δ FUNCTION.
First consider the δ function as an OPERATOR. Then we can write that :
$\delta(x - x')\, f(x) = f(x')\, \delta(x - x')$ since $(x - x') \neq 0$.
Thus the $\int_{-\infty}^{\infty} \delta(x - x') = 1$ illustrated by an AREA = 1. We can then represent the function where
the $\int \delta(x - x')\, f(x) = f(x') = f(x') \int \delta(x - x')\, \partial x$.
This is the δ function in its entirety. Therefore,
$\psi(x') = \int_{-\infty}^{\infty} \frac{\partial P}{\sqrt{2\pi}} e^{iPx'} \left[\int \frac{\partial x}{\sqrt{2\pi}} \psi(x)\, e^{-iPx} \right]$ where we have inserted $\tilde{\psi}(P)$ and simplified to obtain
$= \int \partial x \int \frac{\partial P}{2\pi} e^{iP(x'-x)}\, \psi(x)$. Note that $\frac{\partial P}{2\pi} e^{iP}$ can be simplified by the notation $\triangle = \delta(x - x')$.
So the FOURIER TRANSFORM is thus expressed as the SUM OF PLANE WAVES such that
$\int \frac{\partial P}{2\pi} e^{-iP(x-x')} = \delta(x - x')$
$\int \frac{\partial x}{2\pi} e^{ix(P-P')} = \delta(P - P')$. Obviously we have interchanged (P) with (x). Importantly,
note $\delta(x - x')$ and $\delta(x' - x)$ is a symmetrical function
of its arguments such that the order of (x) and (x') does not matter.

Now we can proceed and describe the notion of a particle on a line .

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9. STATE OF A PARTICLE ON A LINE

We can now describe in detail the meaning of CONSERVATION of Energy using the FOURIER TRANSFORM PAIR.

Classically this is simply defined using (x, P). In VECTOR STATE SPACE we must provide an ORTHOGONAL BASIS OF STATE (normalized or unit vector) so that we realize a LINEAR SUPERPOSITION of the VECTOR (BASIS). In vector state space, the notation

| x >

| P > or

| x P > represents too much information. Recall the formalism

σ_x

σ_y **or**

σ_z.

We therefore cannot know both STATES simulatenously. This is called the HEISENBERG ' s UNCERTAINTY PRINCIPLE. Its one or the other [x or P]. In Classical Physics a PARTICLE can be defined in both (x) and (P);

While in vector state space the particle is described as one or the other. For instance;

$|\psi> = \int \partial(x)\, \psi(x)\, |x>$ **means that the INNER PRODUCT of**

$< x \mid \psi > = \psi(x)$ **which is the projection of the WAVE FUNCTION and**

$\psi^*(x)\, \psi(x) = P(x)$ **is the probability that (x) is at P (x). (* VERY CLEAR AND NEAT *)**

Note that $< x' \mid x > = 0$ **unless (x ' = x).**

But now we are relating the density matrix for the INNER PRODUCT or PURE STATE. So,

$< x' \mid x > = 1 = \delta(x - x')$ **(* Recall that the Integral of the** δ **function is = 1 *)**

So the TOATL PROBABILITY = 1.

We have always described KETS as $\psi(x)$ **and BRAS as** $\psi^*(x)$**. That is** $|\ \psi(x) >$ **and** $< \psi^*(x) \mid$ **respectively. Thus, we can say ,**

$< \psi \mid \phi > = \delta(x - x')$ **and represented as**

$\psi^*(x)\, \phi(x)$ **which when expanded is written as**

$\int \psi^*(x)\, \phi(x)\, \partial x$

If a particle is moving in 2 D, you cannot simulatenously know the x_coordinate of POSITION and the y_coordinate of MOMENTUM. This is the purpose of the FOURIER TRANSFORM PAIR.

From the formalism of functions of vector states, we expect an EIGEN VALUE acting on an EIGEN STATE to give the same EIGEN VECTOR. So an OPERATOR (X) acting on a WAVE FUNCTION gives a number (X) times the wave function as shown below;

$X\ \delta(x - x_o) = X_o\ \delta(x - x_o)$ where $\delta(x - x_o)$ is the Eigen Vector and X_o is the Eigen Value.

This is similar to

$X\ \psi(x) = x\ \psi(x)$ and completes the formalism for (x) or POSITION. What about (P) or MOMENTUM? First, note that (P) is an OBSERVABLE $(REAL)$ and is CONSERVED. It is defined by the HERMITIAN OPERATOR $-i\frac{\partial}{\partial x}$.

So $(P) = -i\hbar\frac{\partial}{\partial x}$. Thus,

$P\ \psi(x) = -i\hbar\frac{\partial \psi}{\partial x}$. Now if we apply the Hermitian operation we obtain

$\frac{\partial \psi}{\partial x} = +i\frac{P_o}{\hbar}\psi(x)$. Recall that this is a partial derivative operation too. Then,

$\psi_{(P)}(x) = e^{i\frac{P_o}{\hbar}(x)}$ means that the function of the wave

is equal to the distance (x) has to move to change to 2π. In other words,

$\frac{P_o}{\hbar}\lambda = 2\pi$ or $\lambda = \frac{2\pi\hbar}{P_o}$. This a clear relation between MOMENTUM (P_o) and WAVELENGTH (λ).

We can thus proceed and evaluate the INNER PRODUCT between two MOMENTUM STATES for $|P_o>$.

$|P_o>$ is represented by

$\psi_{P_o}(x) = e^{iP_o(x)}$. It follows that the complex conjugation of the

$<P'|P> = \int_{-\infty}^{\infty} e^{-iP_o(x)}\, e^{iP(x)}\, \partial x = \int e^{i\,x\,(P - P')}\, \partial x$ and thus factorized. Therefore,

If $P = P'$ then the Integral of 1 becomes infinite. That is $\int_{-\infty}^{\infty} 1\ \partial x = \infty$. The point is,

In order to determine the NUMERICAL COEFFICIENT,

we have to refer back to the δ function and substitute to obtain

$\int e^{i\,x\,(P - P')} = \delta(P - P')$ and since the the INNER PRODUCT of the δ function is REAL,

we have to divide $e^{i\,xP}$ by the $\sqrt{2\pi}$ to obtain :

$\psi_{P_o}(x) = \frac{e^{i\,xP_o}}{\sqrt{2\pi}}$. This is the WAVE FUNCTION of the particle with MOMENTUM (P).

Thus, the WAVE FUNCTION of MOMENTUM can be illustrated as:

$<P|\psi> = \tilde{\psi}(P)$ whereas for POSITION is $<x|\psi> = \psi(x)$. $(*\ \tilde{\psi}$ implies an OBSERVABLE $*)$ So

the WAVE FUNCTION of POSITION can be illustrated as:

$\psi_{x_o}(P) = \frac{e^{-i\,Px_o}}{\sqrt{2\pi}}$ and the solution is the

$\int \frac{e^{-i\,Px}}{\sqrt{2\pi}}\psi(x)\,\partial x$ which is equivalent to the wave function of the FourierTransform. Thus,

$\tilde{\psi}(P) = \psi(x)$ which is exactly the wave function of FourierTransform.

Therefore, the PROBABILITY of finding the wave function of momentum $\tilde{\psi}(P)$ is :

$\mathbb{P}(P) = \tilde{\psi}^*_{(P)}\,\psi(P)$ which is the square of the momentum

(INNER PRODUCT). This is similar to transposing from σ_Z to σ_x Vector State.So,

If $(P) = -i\frac{\partial}{\partial x}$ then

$(x) = +i\frac{\partial}{\partial P}$

Note that TIME, TEMPERATURE, PRESSURE are not a Vector State Operators.

An Electromagnetic Field and Angular Momentum are good examples of a CONJUGATE VARIABLE.

Any equation of the form $\frac{\partial \psi}{\partial t} = -c \frac{\partial \psi}{\partial x}$ has the function

$\psi(x - ct)$. This solves the SCHRODINGER TIME_DEPENDENT EQUATION.

A particle with the COMPLEX FUNCTION $\psi(x - ct)$ moves

rigidly along the x_axis with a uniform velocity (C) ;

where (c) represents the speed of light with the propertty of a + ve and - ve Energy when $\mathcal{H} = cP$;

Thus the PROBABILITY or AVERAGE VALUE or INNER PRODUCT of (x) moves with

velocity (ct). This is the interpretation of the particle in Vector State Space.

Thus, the Hamiltonian representation of the motion of a particle can be written as :

$\frac{\partial \mathcal{H}}{\partial P} = \dot{X}$ and $\frac{\partial \mathcal{H}}{\partial x} = -\dot{P}$

so that $x = c$ and

$-P = 0$ implying the particle is moving at the speed of light while MOMENTUM IS CONSERVED.

(* This is similar to RECOIL, if something hits you. *)

So far we can describe the HAMILTONIAN in terms of Vector State Space where :

T = Kinetic Energy = $\frac{1}{2} mV^2$

P = Momentum = mV and

$\mathcal{H} = \frac{P^2}{2m} = cP$.

Recall that Schrodinger's equation is represented as $i \frac{\partial \psi}{\partial t} = -i\mathcal{H} \mid \psi >$

So $i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left(-i\hbar\right) \frac{\partial}{\partial x} \psi(x)$.

(* P = $-i\frac{\partial}{\partial x}$ and $\left(\frac{\partial}{\partial x}\right)^2 = \frac{\partial^2}{\partial x^2}$;

the square of the 1st derivative is the 2nd derivative *)

But (P) is squared ...So the equation then becomes:

$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left(-i\hbar\right)^2 \frac{\partial^2}{\partial x^2} \psi(x)$ which is described as a particle in one dimension moving along a line. Therefore, we can simplify the equation further to obtain:

$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$. This means that waves with different wavelengths move with different velocities or different momenta move with different velocities.

We conclude by noting that the basic rule for PARTIAL DERIVATIVES is that

$\int F \frac{\partial G}{\partial x} \partial x = -\int \frac{\partial F}{\partial x} G \ \partial x$

where we have shift the factor of the derivative at the cost of -ve sign.

■

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10. UNCERTAINTY PRINCIPLE

Let $(\Delta x)^2 = \int \psi^* \psi\,(x)\, x^2\, \partial x = \langle \psi \mid x^2 \mid \psi \rangle$ and

$(\Delta P)^2 = \int \tilde{\psi}(P)\, \psi(P)\, P^2\, \partial p = \langle \psi \mid P^2 \mid \psi \rangle = -\int \partial x\, \psi^*\,(x)\, \frac{\partial^2 \psi}{\partial x^2} = -\int \partial x\, \psi^*\,(x)\, \frac{\partial}{\partial x}\frac{\partial \psi}{\partial x}$

where $(P) = -i\frac{\partial}{\partial x}$ thus $\left(P^2\right) = -\frac{\partial^2}{\partial x^2}$ and so

$(\Delta P)^2 = +\int \frac{\partial}{\partial x}\psi^*\,(x)\, \frac{\partial \psi}{\partial x}\, \partial x$ which is a complex conjugate and = REAL amd + ve thus

the INNER PRODUCT of $(P)^2$.

(* Recall the basic rules for Partial Derivatives and integrating by parts *)

Consider the TRIANGULAR INEQUALITY (* the product of the magnitude of two vectors is bigger than the dot product between them and Cosϴ ≠ 1 *)

where $\mid A \mid^2 * \mid B \mid^2 > \mid A * B \mid^2$ for the vectors :

Substituting for $(\Delta x)^2 = \mid A \mid^2$ and

$(\Delta P)^2 = \mid B \mid^2$ We would like to show that

$(\Delta x)^2 * (\Delta P)^2 > \hbar$

We first asssume that Wave Function of (x) or $\psi\,(x)$ is REAL

$\mid A > = X\psi\,(x)$ and $\mid B > = P\psi = -i\frac{\partial \psi}{\partial x}$

If we considerthe INNER PRODUCT of the two vectors $\langle A \mid B \rangle$

and assume *a* REAL VALUE (since absolute values are REAL anyway, so we ignore the − ve signs),

then

$\langle A \mid = \mid A \mid^2 = \int \psi(x)\, \frac{\partial \psi}{\partial x} \int \psi(y)\, \frac{\partial \psi}{\partial y}$ (* noting $\frac{\partial}{\partial x}\psi^2 = 2\psi\frac{\partial}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}\psi^2$ *)

$= \frac{1}{2}\int \frac{\partial \psi^2}{\partial x}\,(x)\, \frac{1}{2}\int \frac{\partial \psi^2}{\partial y}\,(y)$

$= \frac{1}{4}\int \frac{\partial \psi^2}{\partial x}\,(x) \int \frac{\partial \psi^2}{\partial x}\,(y)\,.$

We can the INTEGRATE BY PARTS by shifting the derivatives keeping in mind we are expecting an Absolute Value.

So, we have

$\frac{1}{4}\int \frac{\partial x}{\partial x}\psi^2 \int \frac{\partial y}{\partial y}\psi^2$. (* noting $\frac{\partial x}{\partial x} = 1$ = DERIVATIVE OF AN IDENTITY *)

We know that the $\int \psi^2 = 1$

which is the Integral of probability of all space if the wave function is normalized. Therefore, each of the Integrals = 1 when the state vectors are UNITY. We are now left with the value $\frac{1}{4}$.

Thus, $(\triangle x)^2 * (\triangle P)^2 > \frac{1}{4}$. Simplifying we have

$(\triangle x) * (\triangle P) > \frac{1}{2}$ and

since $(P) = -i\frac{\partial}{\partial x}$ meaning (x) and (P) are NOT inverse to each other

(because their product cannot be a pure number) $\frac{P}{\hbar}$ must be inverse to (x). So ,

$(\triangle x) * \frac{\triangle P}{\hbar} > \frac{1}{2} = (\triangle x) * (\triangle P) > \frac{\hbar}{2}$. (* MIMIMUM UNCERTAINTY *)

This is the the Uncertainty Princple and is TRUE for all SPACE.

We can now show that the TIME DERIVATIVE of MOMENT is equivalent to FORCE

where $\dot{P} = F = -i \frac{\partial}{\partial x}$. Consider the Schrodinger Wave Equation $i \frac{\partial \psi}{\partial t} = -i\mathcal{H} \mid \psi >$

Then, considering a particle moving in one Dimension along a line, we can show that

$i \frac{\partial \psi(x_t)}{\partial t} = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} \psi(x_t) + V(x) \psi(x_t)$

which is the Schrodinger ' s representation of how the Wave Function changes with TIME

We can then get rid of the (i) on the left_hand

side by multiplying both sides by (-i) and simplifying to obtain :

$\frac{\partial \dot{\psi}(x_t)}{\partial t} = \frac{i}{2m} \frac{\partial^2}{\partial x^2} \psi(x_t) - i V(x) \psi(x_t)$ since $-i * i = 1$. Therefore,

$\frac{\partial \dot{\psi}^*(x_t)}{\partial t} = -\frac{i}{2m} \frac{\partial^2}{\partial x^2} \psi^*(x_t) + i V(x) \psi^*(x_t)$ which is a complex conjugate.

We thus have both the Schrodinger ' s Equations.

Moving forward, the VELCOCITY OF A WAVE can be represented as :

$\frac{\partial}{\partial t} < x > = \frac{\partial}{\partial t} \int \psi^*(x) \psi(x)$

$= \int \frac{\partial \psi^*}{\partial t}(x) \psi + \int \psi^*(x) \dot{\psi}$

$= \int \dot{\psi}^*(x) \psi + \int \psi^*(x) \dot{\psi}$ which is a complex conjugate and (x) is REAL

So to get $< x >$

we take twice the REAL part which is 2 $\int \psi^*(x) \dot{\psi}$ since the imaginary part cancels.

We then subtitute with the Schrodinger Equation

$\frac{\partial \dot{\psi}(x_t)}{\partial t} = \frac{i}{2m} \frac{\partial^2}{\partial x^2} \psi(x_t) - i V(x) \psi(x_t)$ to obtain :

$\frac{\partial}{\partial t} < x > = 2 \int \psi^*(x) \left[\frac{i}{2m} \frac{\partial^2}{\partial x^2} \psi - i V(x) \psi \right]$ noting that V(x) is imaginary .So

we cancel $(-V(x) \psi)$ and the result

$= 2 \int \psi^*(x) \left[\frac{i}{2m} \frac{\partial^2}{\partial x^2} \psi \right]$

We then must Integrate by Parts. The partial derivative operation yields :

$= -2 \int \psi^*(x) \left[\frac{i}{2m} \frac{\partial}{\partial x} \frac{\partial \psi}{\partial x} \right]$ (* derivative with respect to x without changing time *)

$= -2 \frac{i}{2m} \int \psi^* \left[\frac{\partial \psi}{\partial x} \right] - \frac{i}{2m}(x) \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x}$

$(* \frac{\partial x}{\partial x} = 1$ and we must change signs $*)$ The imaginary part cancels which then

$= -2 \frac{i}{2m} \int \psi^* \left[\frac{\partial \psi}{\partial x} \right] = -\frac{i}{m} \int \psi^* \left[\frac{\partial \psi}{\partial x} \right]$. We know that $-i \frac{\partial}{\partial x} = P = < \psi \mid P \mid \psi >$

thus

$= < \psi \left| \frac{P}{m} \right| \psi >$ which is the INNER PRODUCT of (P) divided by (m).

We have so far proved that

$\frac{\partial <x>}{\partial t} = \frac{<P>}{m}$ and is the Classical relation between Velocity and Momentum

where m $\frac{\partial <x>}{\partial t} = <P>$ implies Mass * Velocity = Momentum.

That is one_half of Classical Mechanics $\frac{\partial x}{\partial t}$.

The othe half is $\frac{\partial P}{\partial t} = F$; which is the TIME DERIVATIVE OF MOMENTUM.

Note that

$\frac{\partial P}{\partial t} = \dot{P} = F = -\frac{\partial V}{\partial x}$ meaning that the time derivative of momentum is equivalent FORCE.

$$\frac{\partial}{\partial t} <P> = \frac{\partial}{\partial t} \int \psi^*(xt)\, \frac{\partial}{\partial x} \psi(xt)\, (-i)$$

$$= -i \int \dot{\psi}^* \frac{\partial \psi}{\partial x} - i \int \psi^* \frac{\partial \psi}{\partial x} \dot{\psi}$$

$$= -i \int \dot{\psi}^* \frac{\partial \psi}{\partial x} + i \int \frac{\partial \psi^*}{\partial x} \psi^* \dot{\psi}.$$ We can cancel the imaginary part so then

$$= +i \int \frac{\partial \psi^*}{\partial x} \psi^* \dot{\psi}$$

Observe that the INNER PRODUCT of $<P>$ has a REAL and must be squared. We then substitute with the Schrodinger Equation to obtain :

$$= i \int \frac{\partial \psi^*}{\partial x} \left[(-i) \frac{1}{2m} \frac{\partial^2 x}{\partial x^2} - V(x) \psi \right].$$

where $\left[(-i) \frac{1}{2m} \frac{\partial^2 x}{\partial x^2} \right]$ is imaginary and cancels . Thus,

$$= \int \frac{\partial \psi^*}{\partial x} V(x) \psi.$$

$\left(\frac{-1}{2m} \frac{\partial^2 x}{\partial x^2} \right)$ can then be Integrated by parts and results in an imaginary number. We are now left with the equation $\int \frac{\partial \psi^*}{\partial x} V(x) \psi$ which must be complex conjugated as :

$$= \int \frac{\partial \psi^*}{\partial x} V(x) \psi + \int \psi^* V(x) \frac{\partial \psi}{\partial x} = \text{REAL}$$

$$= \int V(x) \left[\frac{\partial \psi^*}{\partial x} \psi + \psi^* \frac{\partial \psi}{\partial x} \right]$$ which is the derivative $\frac{\partial \psi^*}{\partial x} \psi$

$$= \int V(x) \left[\frac{\partial \psi^*}{\partial x} \psi \right].$$ If we now Integrate by parts,

the result $= -\int \psi^* \psi \frac{\partial V}{\partial x}$ rewritten as $-\int \psi^* \frac{\partial V}{\partial x} \psi$

thus

$= <\psi^* \frac{\partial V}{\partial x} \psi>$ which is the INNER PRODUCT of $\frac{\partial V}{\partial x}$.

We have so far proved that

$\frac{\partial}{\partial t} <P> = -\frac{\partial V}{\partial x}$ in the sense of the AVERAGE VALUE in a smooth POTENTIAL in PHASE SPACE.

This means the WAVE FUNCTION is NOT disrupted or otherwise there is scattering.

In conclusion, after the painful procedure but worthy,

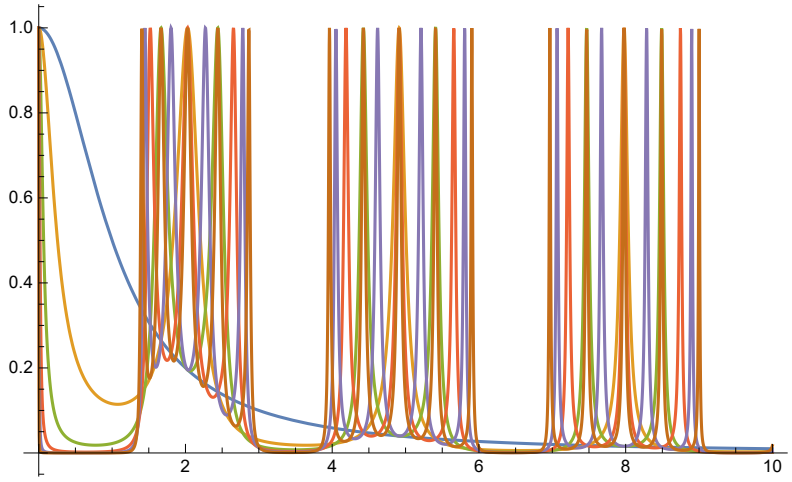
Below are the Wolfram Language CODE and INPUT to :

(a) verify the calculation of MINIMUM UNCERTAINTY of a Radial Wave_Function of the Hydrogen Atom.

```
(* First Compute the energy eigenvalue from the differential equation *)
(* The energy is independent of the orbital quantum number l *)
ψ[n_, l_, r_] := Sqrt[(n - l - 1)!/(n + l)!] e^(-r/n) (2 r/n)^l 2/n^2 LaguerreL[n - l - 1, 2 l + 1, 2 r/n]
energy[w_, r_, n_, l_] := Simplify[1/w (D[w, r, r] + 2/r D[w, r] - l (l + 1)/r^2 w + 2/r w)]
Solve[(energy[ψ[n, l, r], r, n, l] - ε // FullSimplify) == 0, ε]
Out[a] := {{ε → 1/n^2}} and
It is REAL and + ve
```

(b) Visualize Light Amplitude transmission through *n* layers of glass.

```
Plot[Evaluate[Table[1/(1 + k^2 ChebyshevU[n, Cos[k] + Sinc[k]]^2), {n, 0, 5}]],
 {k, 0, 10}, PlotRange → All]
Out[b] :
```



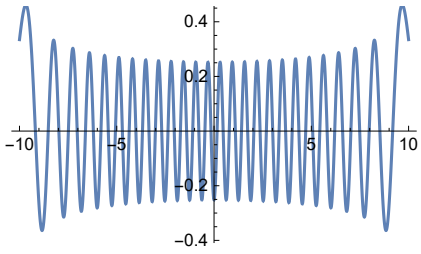
(c) VISUALIZE and VERIFY the Quantum harmonic oscillator Wave Functions

```
(*
Input Wave Function
 Plot
 Normalize
 Compute the expectation value of x^2
 Verify that Momentum and Position wave
 functions for a Harmonic Oscillator have the same form *)
```

$$\psi[n\_, x\_] := \frac{\text{HermiteH}[n, x]\, e^{-\frac{x^2}{2}}}{\sqrt{2^n\, n!\, \sqrt{\pi}}}$$

```
Plot[ψ[50, x], {x, -10, 10}]
∫_-∞^∞ Abs[ψ[10, x]]^2 𝕕x
∫_-∞^∞ x^2 Abs[ψ[10, x]]^2 𝕕x
{FourierTransform[ψ[2, x], x, ω], i^2 ψ[2, x]} // Simplify
(* Notice that (i^2) is the OPERATOR*)
```

Out[832] =



Out[833] = 1

Out[834] = $\frac{21}{2}$

Out[835] = $\left\{\frac{e^{-\frac{\omega^2}{2}}\left(1-2\,\omega^2\right)}{\sqrt{2}\;\pi^{1/4}},\ \frac{e^{-\frac{x^2}{2}}\left(1-2\,x^2\right)}{\sqrt{2}\;\pi^{1/4}}\right\}$

(* Relate the output with $\frac{\partial x}{\partial t}=\frac{P}{m}$ and $\frac{\partial P}{\partial t}=-\frac{\partial V}{\partial x}$ where Position and Momentum are two frequencies in TIME *)

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REFERENCES AND FURTHER READING

The reader who wants information on some phenomena or apparatus is more interested in the information than in knowing who first investigated the phenomena – C.P.Steinmetz

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